Concentration of measures via size biased couplings

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Abstract

Let Y be a nonnegative random variable with mean μ and finite positive variance σ^2, and let Y^s, defined on the same space as Y, have the Y size biased distribution, that is, the distribution characterized by

$$E[Yf(Y)] = \mu Ef(Y^s) \quad \text{for all functions } f \text{ for which these expectations exist.}$$

Under a variety of conditions on the coupling of Y and Y^s, including combinations of boundedness and monotonicity, concentration of measure inequalities such as

$$P\left(\frac{Y-\mu}{\sigma} \geq t\right) \leq \exp\left(-\frac{t^2}{2(A+Bt)}\right) \quad \text{for all } t \geq 0$$

hold for some explicit A and B. Examples include the number of relatively ordered subsequences of a random permutation, sliding window statistics including the number of m-runs in a sequence of coin tosses, the number of local maximum of a random function on a lattice, the number of urns containing exactly one ball in an urn allocation model, the volume covered by the union of n balls placed uniformly over a volume n subset of $\mathbb{R}^d$, the number of bulbs switched on at the terminal time in the so called lightbulb process, the number of isolated vertices in the Erdös-Rényi random graph model, and the infinitely divisible and compound Poisson distributions that satisfy a bounded moment generating function condition.

1 Introduction

Size biasing of random variables is essentially sampling them proportional to their size. Of the many contexts in which size biasing appears, perhaps the most well known is the waiting time paradox, so clearly described in Feller [12], Section I.4. Here, a paradox is generated by the fact that in choosing a time interval 'at random' in which to wait for, say buses, it is more likely that an interval with a longer interarrival time is selected. In statistical contexts it has long been known that size biasing may affect a random sample in adverse ways, though at times this same phenomena may also be used to correct for certain biases [21].

In the realm of normal approximation, size biasing finds a place in Stein's method (see, for instance, [31] and [2]) alongside the exchangeable pair technique. The areas of application of these two techniques are somewhat complementary, with size biasing useful for the approximation of distributions of nonnegative random variables such as counts, and the exchangeable pair for mean zero variates. Though Stein's method has been used mostly for assessing the accuracy of normal approximation, recently related ideas have been proved to be successful in deriving concentration of measure inequalities, that is, deviation inequalities of the form $P(|Y - E(Y)| \geq t\sqrt{\mathrm{Var}(Y)})$, where typically one seeks bounds that decay exponentially in t; for a guide to the literature on the concentration of measures, see [20] for a detailed overview. Regarding the

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use of techniques related to Stein's method to prove such inequalities, Raič obtained large deviation bounds for certain graph related statistics in [28] using the Cramér transform and Chatterjee [7] derived Gaussian and Poisson type tail bounds for Hoeffding's combinatorial CLT and the net magnetization in the Curie-Weiss model in statistical physics in [7]. While the first paper employs the Stein equation, the later applies constructions which are related to the exchangeable pair in Stein's method (see [32]).

For a given nonnegative random variable Y with finite nonzero mean μ, recall (see [15], for example) that Y^s has the Y-size biased distribution if

$$E[Yf(Y)] = \mu E[f(Y^s)] \quad \text{for all functions } f \text{ for which these expectations exist.} \tag{1}$$

Motivated by the complementary connections that exist between the exchangeable pair method and size biasing in Stein's method, we prove the following theorem that shows the parallel persists in the area of concentration of measures, and that size biasing can be used to derive one sided deviation results for nonnegative variables Y that can be closely coupled to a variable Y^s with the Y size biased distribution. Our first result requires the coupling to be bounded.

Theorem 1.1. *Let Y be a nonnegative random variable with mean and variance μ and σ^2 respectively, both finite and positive. Suppose there exists a coupling of Y to a variable Y^s having the Y-size bias distribution which satisfies $|Y^s - Y| \le C$ for some $C > 0$ with probability one.*

If $Y^s \ge Y$ with probability one, then

$$P\left(\frac{Y-\mu}{\sigma} \le -t\right) \le \exp\left(-\frac{t^2}{2A}\right) \quad \text{for all } t > 0, \text{ where } A = C\mu/\sigma^2. \tag{2}$$

If the moment generating function $m(\theta) = E(e^{\theta Y})$ is finite at $\theta = 2/C$, then

$$P\left(\frac{Y-\mu}{\sigma} \ge t\right) \le \exp\left(-\frac{t^2}{2(A+Bt)}\right) \quad \text{for all } t > 0, \text{ where } A = C\mu/\sigma^2 \text{ and } B = C/2\sigma. \tag{3}$$

The monotonicity hypothesis for inequality (2), that $Y^s \ge Y$, is natural since Y^s is stochastically larger than Y. Therefore there always exists a coupling for which $Y^s \ge Y$. There is no guarantee, however, that for such a monotone coupling, the difference $Y^s - Y$ is bounded. For (3) we note that the moment generating function is finite everywhere when Y is bounded. In typical examples the variable Y is indexed by n, and the ones we consider have the property that the ratio μ/σ^2 remains bounded as $n \to \infty$, and C does not depend on n. In such cases the bound in (2) decreases at rate $\exp(-ct^2)$ for some $c > 0$, and if $\sigma \to \infty$ as $n \to \infty$, the bound in (3) is of similar order, asymptotically.

Examples covered by Theorem 1.1 are given in Section 4, and include the number of relatively ordered subsequences of a random permutation, sliding window statistics including the number of m-runs in a sequence of coin tosses, the number of local maximum of a random function on the lattice, the number of urns containing exactly one ball in the uniform urn allocation model, the volume covered by the union of n balls placed uniformly over a volume n subset of $\mathbb{R}^d$, and the number of bulbs switched on at the terminal time in the so called lightbulb problem.

In Section 5 we also consider cases where the coupling of Y^s and Y is unbounded, handled on a somewhat case by case basis. Our examples include the number of isolated vertices in the Erdös-Rényi random graph model, and some infinitely divisible and compound Poisson distributions. As Theorem 1.1 shows, additional information is available when the coupling is monotone; this condition holds for the m runs, lightbulb and isolated vertices examples, as well as the infinitely divisible and compound Poisson distributions considered.

A number of results in Stein's method for normal approximation rest on the fact that if a variable Y of interest can be closely coupled to some related variable, then the distribution of Y is close to normal. An advantage, therefore, of the Stein method is that dependence can be handled in a direct manner, by the construction of couplings on the given collection of random variables related to Y. In [28] and [7],

ideas related to Stein's method were used to obtain concentration of measure inequalities in the presence of dependence.

Of the two, the technique used by Chatterjee in [7], based on Stein's exchangeable pair [32], is the one closer to the approach taken here. We say Y, Y' is a λ-Stein pair if these variables are exchangeable and satisfy the linearity condition

$$E(Y - Y'|Y) = \lambda Y \quad \text{for some } \lambda \in (0, 1). \tag{4}$$

The λ-Stein pair is clearly the special case of the more general identity

$$E(F(Y, Y')|Y) = f(Y) \quad \text{for some antisymmetric function } F,$$

specialized to $F(Y, Y') = Y - Y'$ and $f(y) = \lambda y$. Chatterjee in [7] considers a pair of variables satisfying this more general identity, and, with

$$\Delta(Y) = \frac{1}{2}E((f(Y) - f(Y'))F(Y, Y')|Y),$$

obtains a concentration of measure inequality for Y under the assumption that $\Delta(Y) \leq Bf(Y) + C$ for some constants B and C.

For normal approximation, as seems to be the case here also, the areas in which pair couplings such as (4) apply, and those for which size bias coupling of Theorem 1.1 succeed, appear to be somewhat disjoint. In particular, (4) seems to be more suited to variables which arise with mean zero, while the size bias couplings work well for variables, such as counts, which are necessarily nonnegative. Indeed, for the problems we consider, there appears to be no natural way by which to find exchangeable pairs satisfying the conditions of [7]. On the other hand, the size bias couplings applied here are easy to obtain.

After proving Theorem 1.1 in Section 2, in Section 3 we review the methods in [15] for the construction of size bias couplings in the presence of dependence, and then move to the examples already mentioned.

2 Proof of the main result

In the sequel we make use of the following inequality, which depends on the convexity of the exponential function;

$$\frac{e^y - e^x}{y - x} = \int_0^1 e^{ty+(1-t)x}dt \leq \int_0^1 (te^y + (1-t)e^x)dt = \frac{e^y + e^x}{2} \quad \text{for all } x \neq y. \tag{5}$$

We now move to the proof of Theorem 1.1.

Proof. Recall Y^s is given on the same space as Y, and has the Y size biased distribution. By (5), for all $\theta \in \mathbb{R}$, since $|Y^s - Y| \leq C$,

$$|e^{\theta Y^s} - e^{\theta Y}| \leq \frac{1}{2}|\theta(Y^s - Y)|(e^{\theta Y^s} + e^{\theta Y}) \leq \frac{C|\theta|}{2}(e^{\theta Y^s} + e^{\theta Y}). \tag{6}$$

Recalling that if the moment generating function $m(\theta) = E[e^{\theta Y}]$ exists in an open interval containing θ then we may differentiate under the expectation, we obtain

$$m'(\theta) = E[Ye^{\theta Y}] = \mu E[e^{\theta Y^s}]. \tag{7}$$

To prove (2), let $\theta < 0$ and note that since the coupling is monotone $\exp(\theta Y^s) \leq \exp(\theta Y)$. Now (6) yields

$$e^{\theta Y} - e^{\theta Y^s} \leq C|\theta|e^{\theta Y}.$$

Since $Y \geq 0$ the moment generating function $m(\theta)$ exists for all $\theta < 0$, so taking expectation and rearranging yields

$$Ee^{\theta Y^s} \geq (1 - C|\theta|)Ee^{\theta Y} = (1 + C\theta)E(e^{\theta Y}),$$

and now, by (7),

$$m'(\theta) \geq \mu(1 + C\theta)m(\theta) \quad \text{for all } \theta < 0. \tag{8}$$

To consider standardized deviations of Y, that is, deviations of $|Y - \mu|/\sigma$, let

$$M(\theta) = Ee^{\theta(Y-\mu)/\sigma} = e^{-\theta\mu/\sigma}m(\theta/\sigma). \tag{9}$$

Now rewriting (8) in terms of $M(\theta)$, we obtain for all $\theta < 0$,

$$\begin{aligned} M'(\theta) &= -(\mu/\sigma)e^{-\theta\mu/\sigma}m(\theta/\sigma) + e^{-\theta\mu/\sigma}m'(\theta/\sigma)/\sigma \\ &\geq -(\mu/\sigma)e^{-\theta\mu/\sigma}m(\theta/\sigma) + (\mu/\sigma)e^{-\theta\mu/\sigma}\left(1 + \frac{C\theta}{\sigma}\right)m(\theta/\sigma) \\ &= (\mu/\sigma^2)C\theta M(\theta). \end{aligned} \tag{10}$$

Since $M(0) = 1$, by (10)

$$-\log M(\theta) = \int_\theta^0 \frac{M'(s)}{M(s)}ds \geq \int_\theta^0 \frac{C\mu s}{\sigma^2}ds = -\frac{C\mu\theta^2}{2\sigma^2},$$

so exponentiation gives us

$$M(\theta) \leq \exp\left(\frac{C\mu\theta^2}{2\sigma^2}\right) \quad \text{when } \theta < 0.$$

Hence for a fixed $t > 0$, for all $\theta < 0$,

$$\begin{aligned} P\left(\frac{Y-\mu}{\sigma} \leq -t\right) = P\left(\theta\left(\frac{Y-\mu}{\sigma}\right) \geq -\theta t\right) &= P\left(e^{\theta\left(\frac{Y-\mu}{\sigma}\right)} \geq e^{-\theta t}\right) \\ &\leq e^{\theta t}M(\theta) \leq \exp\left(\theta t + \frac{C\mu\theta^2}{2\sigma^2}\right). \end{aligned} \tag{11}$$

Substituting $\theta = -t\sigma^2/(C\mu)$ into (11) completes the proof of (2).

Moving on to the proof of (3), taking expectation in (6) with $\theta > 0$, we obtain

$$Ee^{\theta Y^s} - Ee^{\theta Y} \leq \frac{C\theta}{2}\left(Ee^{\theta Y^s} + Ee^{\theta Y}\right),$$

so in particular, when $0 < \theta < 2/C$,

$$E[e^{\theta Y^s}] \leq \left(\frac{1 + C\theta/2}{1 - C\theta/2}\right)E[e^{\theta Y}]. \tag{12}$$

As $m(2/C) < \infty$, (7) applies and (12) yields

$$m'(\theta) \leq \mu\left(\frac{1 + C\theta/2}{1 - C\theta/2}\right)m(\theta) \quad \text{for all } 0 < \theta < 2/C. \tag{13}$$

Now letting $\theta \in (0, 2\sigma/C)$, from (9), $M(\theta)$ is differentiable for all $\theta < 2\sigma/C$ and (13) yields,

$$\begin{aligned} M'(\theta) &= -(\mu/\sigma)e^{-\theta\mu/\sigma}m(\theta/\sigma) + e^{-\theta\mu/\sigma}m'(\theta/\sigma)/\sigma \\ &\leq -(\mu/\sigma)e^{-\theta\mu/\sigma}m(\theta/\sigma) + (\mu/\sigma)e^{-\theta\mu/\sigma}\left(\frac{1+C\theta/(2\sigma)}{1-C\theta/(2\sigma)}\right)m(\theta/\sigma) \\ &= (\mu/\sigma)e^{-\theta\mu/\sigma}m(\theta/\sigma)\left(\left(\frac{1+C\theta/(2\sigma)}{1-C\theta/(2\sigma)}\right)-1\right) \\ &= (\mu/\sigma^2)\left(\frac{C\theta}{1-C\theta/(2\sigma)}\right)M(\theta). \end{aligned}$$

Dividing by $M(\theta)$ we may rewrite the inequality as

$$\frac{d}{d\theta}\log M(\theta) \leq (\mu/\sigma^2)\left(\frac{C\theta}{1-C\theta/(2\sigma)}\right).$$

Noting that $M(0) = 1$, setting $A = C\mu/\sigma^2$ and $B = C/(2\sigma)$, integrating we obtain

$$\log M(\theta) = \int_0^\theta \frac{d}{ds}\log M(s)\,ds \leq (\mu/\sigma^2)\int_0^\theta\left(\frac{Cs}{1-Bs}\right)ds = (\mu/\sigma^2)\frac{C\theta^2}{2(1-B\theta)} = \frac{A\theta^2}{2(1-B\theta)}.$$

Hence, for $t > 0$,

$$P\left(\frac{Y-\mu}{\sigma} \geq t\right) = P\left(\theta(\frac{Y-\mu}{\sigma}) \geq \theta t\right) = P\left(e^{\theta\left(\frac{Y-\mu}{\sigma}\right)} \geq e^{\theta t}\right) \leq e^{-\theta t}M(\theta) \leq e^{-\theta t}\exp\left(\frac{A\theta^2}{2(1-B\theta)}\right).$$

Noting that $\theta = t/(A+Bt)$ lies in $(0, 2\sigma/C)$ for all $t > 0$, substituting this value yields the bound

$$P\left(\frac{Y-\mu}{\sigma} \geq t\right) < \exp\left(-\frac{t^2}{2(A+Bt)}\right) \quad \text{for all } t > 0,$$

completing the proof. □ □

3 Construction of size bias couplings

In this section we will review the discussion in [15] which gives a procedure for a construction of size bias couplings when Y is a sum; the method has its roots in the work of Baldi et al. [1]. The construction depends on being able to size bias a collection of nonnegative random variables in a given coordinate, as described in the following definition. Letting F be the distribution of Y, first note that the characterization (1) of the size bias distribution F^s is equivalent to the specification of F^s by its Radon Nikodym derivative

$$dF^s(x) = \frac{x}{\mu}dF(x). \tag{14}$$

Definition 3.1. *Let $\mathcal{A}$ be an arbitrary index set and let $\{X_\alpha : \alpha \in \mathcal{A}\}$ be a collection of nonnegative random variables with finite, nonzero expectations $EX_\alpha = \mu_\alpha$ and joint distribution $dF(\mathbf{x})$. For $\beta \in \mathcal{A}$, we say that $\mathbf{X}^\beta = \{X_\alpha^\beta : \alpha \in \mathcal{A}\}$ has the $\mathbf{X}$ size bias distribution in coordinate β if $\mathbf{X}^\beta$ has joint distribution*

$$dF^\beta(\mathbf{x}) = x_\beta dF(\mathbf{x})/\mu_\beta.$$

Just as (14) is related to (1), the random vector $\mathbf{X}^\beta$ has the $\mathbf{X}$ size bias distribution in coordinate β if and only if

$$E[X_\beta f(\mathbf{X})] = \mu_\beta E[f(\mathbf{X}^\beta)] \quad \text{for all functions } f \text{ for which these expectations exist.}$$

Now letting $f(\mathbf{X}) = g(X_\beta)$ for some function g one recovers (1), showing that the β^{th} coordinate of $\mathbf{X}^\beta$, that is, X_β^β, has the X_β size bias distribution.

The factorization

$$P(\mathbf{X} \in d\mathbf{x}) = P(\mathbf{X} \in d\mathbf{x}|X_\beta = x)P(X_\beta \in dx)$$

of the joint distribution of $\mathbf{X}$ suggests a way to construct $\mathbf{X}$. First generate X_β, a variable with distribution $P(X_\beta \in dx)$. If $X_\beta = x$, then generate the remaining variates $\{X_\alpha^\beta, \alpha \neq \beta\}$ with distribution $P(\mathbf{X} \in d\mathbf{x}|X_\beta = x)$. Now, by the factorization of $dF(\mathbf{x})$, we have

$$dF^\beta(\mathbf{x}) = x_\beta dF(\mathbf{x})/\mu_\beta = P(\mathbf{X} \in d\mathbf{x}|X_\beta = x)x_\beta P(X_\beta \in dx)/\mu_\beta = P(\mathbf{X} \in d\mathbf{x}|X_\beta = x)P(X_\beta^\beta \in dx). \quad (15)$$

Hence, to generate $\mathbf{X}^\beta$ with distribution dF^β, first generate a variable X_β^β with the X_β size bias distribution, then, when $X_\beta^\beta = x$, generate the remaining variables according to their original conditional distribution given that the β^{th} coordinate takes on the value x.

Definition 3.1 and the following proposition from Section 2 of [15] will be applied in the subsequent constructions; the reader is referred there for the simple proof.

Proposition 3.1. *Let $\mathcal{A}$ be an arbitrary index set, and let $\mathbf{X} = \{X_\alpha, \alpha \in \mathcal{A}\}$ be a collection of nonnegative random variables with finite means. For any subset $B \subset \mathcal{A}$, set*

$$X_B = \sum_{\beta \in B} X_\beta \quad \textit{and} \quad \mu_B = EX_B.$$

Suppose $B \subset \mathcal{A}$ with $0 < \mu_B < \infty$, and for $\beta \in B$ let $\mathbf{X}^\beta$ have the $\mathbf{X}$-size biased distribution in coordinate β as in Definition 3.1. If $\mathbf{X}^B$ has the mixture distribution

$$\mathcal{L}(\mathbf{X}^B) = \sum_{\beta \in B} \frac{\mu_\beta}{\mu_B} \mathcal{L}(\mathbf{X}^\beta),$$

then

$$EX_B f(\mathbf{X}) = \mu_B E f(\mathbf{X}^B)$$

for all real valued functions f for which these expectations exist. Hence, for any $A \subset \mathcal{A}$, if f is a function of $X_A = \sum_{\alpha \in A} X_\alpha$ only,

$$EX_B f(X_A) = \mu_B E f(X_A^B) \quad \textit{where} \quad X_A^B = \sum_{\alpha \in A} X_\alpha^B. \quad (16)$$

Taking $A = B$ in (16) we have $EX_A f(X_A) = \mu_A E f(X_A^A)$, and hence X_A^A has the X_A-size biased distribution, as in (1).

In our examples we use Proposition 3.1 and (15) to obtain a variable Y^s with the size bias distribution of Y, where $Y = \sum_{\alpha \in A} X_\alpha$, as follows. First choose a random index $I \in A$ with probability

$$P(I = \alpha) = \mu_\alpha/\mu_A, \quad \alpha \in A.$$

Next generate X_I^I with the size bias distribution of X_I. If $I = \alpha$ and $X_\alpha^\alpha = x$, generating $\{X_\beta^\alpha : \beta \in A \setminus \{\alpha\}\}$ using the (original) conditional distribution

$$P(X_\beta, \beta \neq \alpha | X_\alpha = x),$$

the sum $Y^s = \sum_{\alpha \in A} X_\alpha^I$ has the Y size biased distribution.

4 First applications: bounded couplings

We now consider the application of Theorem 1.1 to derive concentration of measure results for the number of relatively ordered subsequences of a random permutation, the number of m-runs in a sequence of coin tosses, the number of local extrema on a graph, the number of nonisolated balls in an urn allocation model, the covered volume in binomial coverage process, and the number of bulbs lit at the terminal time in the so called lightbulb process. Without further mention we will use the fact that when (2) and (3) hold for some A and B then they also hold when these values are replaced by any larger ones, which may also be denoted by A and B.

4.1 Relatively ordered sub-sequences of a random permutation

For $n \geq m \geq 3$, let π and τ be permutations of $\mathcal{V} = \{1, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively, and let

$$\mathcal{V}_\alpha = \{\alpha, \alpha + 1, \ldots, \alpha + m - 1\} \quad \text{for } \alpha \in \mathcal{V},$$

where addition of elements of $\mathcal{V}$ is modulo n. We say the pattern τ appears at location $\alpha \in \mathcal{V}$ if the values $\{\pi(v)\}_{v \in \mathcal{V}_\alpha}$ and $\{\tau(v)\}_{v \in \mathcal{V}_1}$ are in the same relative order. Equivalently, the pattern τ appears at α if and only if $\pi(\tau^{-1}(v) + \alpha - 1), v \in \mathcal{V}_1$ is an increasing sequence. When $\tau = \iota_m$, the identity permutation of length m, we say that π has a rising sequence of length m at position α. Rising sequences are studied in [6] in connection with card tricks and card shuffling.

Letting π be chosen uniformly from all permutations of $\{1, \ldots, n\}$, and X_α the indicator that τ appears at α,

$$X_\alpha(\pi(v), v \in \mathcal{V}_\alpha) = 1(\pi(\tau^{-1}(1) + \alpha - 1) < \cdots < \pi(\tau^{-1}(m) + \alpha - 1)),$$

the sum $Y = \sum_{\alpha \in \mathcal{V}} X_\alpha$ counts the number of m-element-long segments of π that have the same relative order as τ.

For $\alpha \in \mathcal{V}$ we may generate $\mathbf{X}^\alpha = \{X^\alpha_\beta, \beta \in \mathcal{V}\}$ with the $\mathbf{X} = \{X_\beta, \beta \in \mathcal{V}\}$ distribution size biased in direction α, following [13]. Let σ_α be the permutation of $\{1, \ldots, m\}$ for which

$$\pi(\sigma_\alpha(1) + \alpha - 1) < \cdots < \pi(\sigma_\alpha(m) + \alpha - 1),$$

and set

$$\pi^\alpha(v) = \begin{cases} \pi(\sigma_\alpha(\tau(v - \alpha + 1)) + \alpha - 1), & v \in \mathcal{V}_\alpha \\ \pi(v) & v \notin \mathcal{V}_\alpha. \end{cases}$$

In other words π^α is the permutation π with the values $\pi(v), v \in \mathcal{V}_\alpha$ reordered so that $\pi^\alpha(\gamma)$ for $\gamma \in \mathcal{V}_\alpha$ are in the same relative order as τ. Now let

$$X^\alpha_\beta = X_\beta(\pi^\alpha(v), v \in \mathcal{V}_\beta),$$

the indicator that τ appears at position β in the reordered permutation π^α. As π^α and π agree except perhaps for the m values in $\mathcal{V}_\alpha$, we have

$$X^\alpha_\beta = X_\beta(\pi(v), v \in \mathcal{V}_\beta) \quad \text{for all } |\beta - \alpha| \geq m.$$

Hence, as

$$|Y^\alpha - Y| \leq \sum_{|\beta - \alpha| \leq m-1} |X^\alpha_\beta - X_\beta| \leq 2m - 1. \tag{17}$$

we may take $C = 2m - 1$ as the almost sure bound on the coupling of Y^s and Y.

Regarding the mean μ of Y, clearly for any τ, as all relative orders of $\pi(v), v \in \mathcal{V}_\alpha$ are equally likely,

$$EX_\alpha = 1/m! \quad \text{and therefore} \quad \mu = n/m!. \tag{18}$$

To compute the variance, for $0 \le k \le m-1$, let I_k be the indicator that $\tau(1), \ldots, \tau(m-k)$ and $\tau(k+1), \ldots, \tau(m)$ are in the same relative order. Clearly $I_0 = 1$, and for rising sequences, as $\tau(j) = j$, $I_k = 1$ for all k. In general for $0 \le k \le m-1$ we have $X_\alpha X_{\alpha+k} = 0$ if $I_k = 0$, as the joint event in this case demands two different relative orders on the segment of π of length $m-k$ of which both X_α and $X_{\alpha+k}$ are a function. If $I_k = 1$ then a given, common, relative order is demanded for this same length of π, and relative orders also for the two segments of length k on which exactly one of X_α and X_β depend, and so, in total a relative order on $m-k+2k = m+k$ values of π, and therefore

$$EX_\alpha X_{\alpha+k} = I_k/(m+k)! \quad \text{and} \quad \mathrm{Cov}(X_\alpha, X_{\alpha+k}) = I_k/(m+k)! - 1/(m!)^2.$$

As the relative orders of non-overlapping segments of π are independent, now taking $n \ge 2m$, the variance σ^2 of Y is given by

$$\begin{aligned}
\sigma^2 &= \sum_{\alpha\in\mathcal{V}} \mathrm{Var}(X_\alpha) + \sum_{\alpha\neq\beta} \mathrm{Cov}(X_\alpha, X_\beta) \\
&= \sum_{\alpha\in\mathcal{V}} \mathrm{Var}(X_\alpha) + \sum_{\alpha\in\mathcal{V}} \sum_{\beta:1\le|\alpha-\beta|\le m-1} \mathrm{Cov}(X_\alpha, X_\beta) \\
&= \sum_{\alpha\in\mathcal{V}} \mathrm{Var}(X_\alpha) + 2\sum_{\alpha\in\mathcal{V}} \sum_{k=1}^{m-1} \mathrm{Cov}(X_\alpha, X_{\alpha+k}) \\
&= n\mathrm{Var}(X_1) + 2n\sum_{k=1}^{m-1} \mathrm{Cov}(X_1, X_{1+k}) \\
&= n\left(\frac{1}{m!} - \frac{1}{(m!)^2}\right) + 2n\sum_{k=1}^{m-1}\left(\frac{I_k}{(m+k)!} - (\frac{1}{m!})^2\right) \\
&= n\left(\frac{1}{m!}\left(1 - \frac{2m-1}{m!}\right) + 2\sum_{k=1}^{m-1}\frac{I_k}{(m+k)!}\right).
\end{aligned}$$

Clearly $\mathrm{Var}(Y)$ is maximized for the identity permutation $\tau(k) = k, k = 1, \ldots, m$, as $I_m = 1$ for all $1 \le m \le m-1$, and as mentioned, this case corresponds to counting the number of rising sequences. In contrast, the variance lower bound

$$\sigma^2 \ge \frac{n}{m!}\left(1 - \frac{2m-1}{m!}\right) \tag{19}$$

is attained at the permutation

$$\tau(j) = \begin{cases} 1 & j = 1 \\ j+1 & 2 \le j \le m-1 \\ 2 & j = m \end{cases}$$

which has $I_k = 0$ for all $1 \le k \le m-1$. In particular, the bound (3) of Theorem 1.1 holds with

$$A = \frac{2m-1}{1 - \frac{2m-1}{m!}} \quad \text{and} \quad B = \frac{2m-1}{2\sqrt{\frac{n}{m!}\left(1 - \frac{2m-1}{m!}\right)}}.$$

4.2 Local Dependence

The following lemma shows how to construct a collection of variables $\mathbf{X}^\alpha$ having the $\mathbf{X}$ distribution biased in direction α when X_α is some function of a subset of a collection of independent random variables.

Lemma 4.1. *Let $\{C_g, g \in \mathcal{V}\}$ be a collection of independent random variables, and for each $\alpha \in \mathcal{V}$ let $\mathcal{V}_\alpha \subset \mathcal{V}$ and $X_\alpha = X_\alpha(C_g, g \in \mathcal{V}_\alpha)$ be a nonnegative random variable with a nonzero, finite expectation. Then if $\{C_g^\alpha, g \in \mathcal{V}_\alpha\}$ has distribution*

$$dF^\alpha(c_g, g \in \mathcal{V}_\alpha) = \frac{X_\alpha(c_g, g \in \mathcal{V}_\alpha)}{EX_\alpha(C_g, g \in \mathcal{V}_\alpha)} dF(c_g, g \in \mathcal{V}_\alpha)$$

and is independent of $\{C_g, g \in \mathcal{V}\}$, letting

$$X_\beta^\alpha = X_\beta(C_g^\alpha, g \in \mathcal{V}_\beta \cap \mathcal{V}_\alpha, C_g, g \in \mathcal{V}_\beta \cap \mathcal{V}_\alpha^c),$$

the collection $\mathbf{X}^\alpha = \{X_\beta^\alpha, \beta \in \mathcal{V}\}$ has the $\mathbf{X}$ distribution biased in direction α.

Furthermore, with I chosen proportional to EX_α, independent of the remaining variables, the sum

$$Y^s = \sum_{\beta \in \mathcal{V}} X_\beta^I$$

has the Y size biased distribution, and when there exists M such that $X_\alpha \le M$ for all α,

$$|Y^s - Y| \le bM \quad \text{where} \quad b = \max_\alpha |\{\beta : \mathcal{V}_\beta \cap \mathcal{V}_\alpha \neq \emptyset\}|. \tag{20}$$

Proof. By independence, the random variables

$$\{C_g^\alpha, g \in \mathcal{V}_\alpha\} \cup \{C_g, g \notin \mathcal{V}_\alpha\} \quad \text{have distribution} \quad dF^\alpha(c_g, g \in \mathcal{V}_\alpha) dF(c_g, g \notin \mathcal{V}_\alpha).$$

Thus, with $\mathbf{X}^\alpha$ as given, we find

$$\begin{aligned} EX_\alpha f(\mathbf{X}) &= \int x_\alpha f(\mathbf{x}) dF(c_g, g \in \mathcal{V}) \\ &= EX_\alpha \int f(\mathbf{x}) \frac{x_\alpha dF(c_g, g \in \mathcal{V}_\alpha)}{EX_\alpha(C_g, g \in \mathcal{V}_\alpha)} dF(c_g, g \notin \mathcal{V}_\alpha) \\ &= EX_\alpha \int f(\mathbf{x}) dF^\alpha(c_g, g \in \mathcal{V}_\alpha) dF(c_g, g \notin \mathcal{V}_\alpha) \\ &= EX_\alpha E f(\mathbf{X}^\alpha). \end{aligned}$$

That is, $\mathbf{X}^\alpha$ has the $\mathbf{X}$ distribution biased in direction α, as in Definition 3.1.

The claim on Y^s follows from Proposition 3.1, and finally, since $X_\beta = X_\beta^\alpha$ whenever $\mathcal{V}_\beta \cap \mathcal{V}_\alpha = \emptyset$,

$$|Y^s - Y| \le \sum_{\beta : \mathcal{V}_\beta \cap \mathcal{V}_I \neq \emptyset} |X_\beta^I - X_\beta| \le bM.$$

This completes the proof. □ □

4.2.1 Sliding m window statistics

For $n \ge m \ge 1$, let $\mathcal{V} = \{1, \dots, n\}$ considered modulo n, $\{C_g : g \in \mathcal{V}\}$ i.i.d. real valued random variables, and for each $\alpha \in \mathcal{V}$ set

$$\mathcal{V}_\alpha = \{v \in \mathcal{V} : \alpha \le v \le \alpha + m - 1\}.$$

Then for $X : \mathbb{R}^m \to [0, 1]$, say, Lemma 4.1 may be applied to the sum $Y = \sum_{\alpha \in \mathcal{V}} X_\alpha$ of the m-dependent sequence $X_\alpha = X(C_\alpha, \dots, C_{\alpha+m-1})$, formed by applying the function X to the variables in the 'm-window' $\mathcal{V}_\alpha$. As for all α we have $X_\alpha \le 1$ and

$$\max_\alpha |\{\beta : \mathcal{V}_\beta \cap \mathcal{V}_\alpha \neq \emptyset\}| = 2m - 1,$$

we may take $C = 2m - 1$ in Theorem 1.1, by Lemma 4.1.

For a concrete example let Y be the number of m runs of the sequence $\xi_1, \xi_2, \ldots, \xi_n$ of n i.i.d Bernoulli(p) random variables with $p \in (0,1)$, given by $Y = \sum_{i=1}^n X_i$ where $X_i = \xi_i \xi_{i+1} \cdots \xi_{i+m-1}$, with the periodic convention $\xi_{n+k} = \xi_k$. In [30], the authors develop smooth function bounds for normal approximation for the case of 2-runs. Note that the construction given in Lemma 4.1 for this case is monotone, as for any i, letting

$$\xi_j' = \begin{cases} \xi_j & j \notin \{i, \ldots, i+m-1\} \\ 1 & j \in \{i, \ldots, i+m-1\}, \end{cases}$$

the number of m runs of $\{\xi_j'\}_{i=1}^n$, that is $Y^s = \sum_{i=1}^n \xi_i' \xi_{i+1}' \cdots \xi_{i+m-1}'$, is at least Y.

For the mean of Y clearly $\mu = np^m$. For the variance, now letting $n \geq 2m$ and using the fact that non-overlapping segments of the sequence are independent,

$$\begin{aligned} \sigma^2 &= \sum_{i=1}^n \mathrm{Var}(\xi_i \xi_{i+1} \cdots \xi_{i+m-1}) + 2 \sum_{i<j} \mathrm{Cov}(\xi_i \cdots \xi_{i+m-1}, \xi_j \cdots \xi_{j+m-1}) \\ &= np^m(1-p^m) + 2 \sum_{i=1}^n \sum_{j=1}^{m-1} \mathrm{Cov}(\xi_i \cdots \xi_{i+m-1}, \xi_{i+j} \cdots \xi_{i+j+m-1}). \end{aligned}$$

For the covariances,

$$\begin{aligned} \mathrm{Cov}(\xi_i \cdots \xi_{i+m-1}, \xi_{i+j} \cdots \xi_{i+j+m-1}) &= E(\xi_i \cdots \xi_{i+j-1} \xi_{i+j} \cdots \xi_{i+m-1} \xi_{i+m} \cdots \xi_{i+j+m-1}) - p^{2m} \\ &= p^{m+j} - p^{2m}, \end{aligned}$$

and therefore

$$\sigma^2 = np^m \left((1-p^m) + 2 \left(\frac{p - p^m}{1-p} - (m-1)p^m \right) \right) = np^m \left(1 + 2\frac{p-p^m}{1-p} - (2m-1)p^m \right).$$

Hence (2) and (3) of Theorem 1.1 hold with

$$A = \frac{2m-1}{1 + 2\frac{p-p^m}{1-p} - (2m-1)p^m} \quad \text{and} \quad B = \frac{2m-1}{2\sqrt{np^m \left(1 + 2\frac{p-p^m}{1-p} - (2m-1)p^m \right)}}.$$

4.2.2 Local extrema on a lattice

Size biasing the number of local extrema on graphs, for the purpose of normal approximation, was studied in [1] and [13]. For a given graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, let $\mathcal{G}_v = \{\mathcal{V}_v, \mathcal{E}_v\}, v \in \mathcal{V}$, be a collection of isomorphic subgraphs of $\mathcal{G}$ such that $v \in \mathcal{V}_v$ and for all $v_1, v_2 \in \mathcal{V}$ the isomorphism from $\mathcal{G}_{v_1}$ to $\mathcal{G}_{v_2}$ maps v_1 to v_2. Let $\{C_g, g \in \mathcal{V}\}$ be a collection of independent and identically distributed random variables, and let X_v be defined by

$$X_v(C_w, w \in \mathcal{V}_v) = 1(C_v > C_w, w \in \mathcal{V}_v), \quad v \in \mathcal{V}.$$

Then the sum $Y = \sum_{v \in \mathcal{V}} X_v$ counts the number local maxima. In general one may define the neighbor distance d between two vertices $v, w \in \mathcal{V}$ by

$$d(v,w) = \min\{n : \text{there } \exists\ v_0, \ldots, v_n \text{ in } \mathcal{V} \text{ such that } v_0 = v, v_n = w \text{ and } (v_k, v_{k+1}) \in \mathcal{E} \text{ for } k = 0, \ldots, n\}.$$

Then for $v \in \mathcal{V}$ and $r = 0, 1, \ldots$,

$$\mathcal{V}_v(r) = \{w \in \mathcal{V} : d(w,v) \leq r\}$$

is the set of vertices of $\mathcal{V}$ at distance at most r from v. We suppose that the given isomorphic graphs are of this form, that is, that there is some r such that $\mathcal{V}_v = \mathcal{V}_v(r)$ for all $v \in \mathcal{V}$. Then if $d(v_1, v_2) > 2r$, and $(w_1, w_2) \in \mathcal{V}_{v_1} \times \mathcal{V}_{v_2}$, rearranging

$$2r < d(v_1, v_2) \le d(v_1, w_1) + d(w_1, w_2) + d(w_2, v_2)$$

and using $d(v_i, w_i) \le r, i = 1, 2$, yields $d(w_1, w_2) > 0$. Hence,

$$d(v_1, v_2) > 2r \quad \text{implies} \quad \mathcal{V}_{v_1} \bigcap \mathcal{V}_{v_2} = \emptyset, \quad \text{so by (20) we may take} \quad b = \max_v |\mathcal{V}_v(2r)|. \tag{21}$$

For example, for $p \in \{1, 2, \ldots\}$ and $n \ge 5$ consider the lattice $\mathcal{V} = \{1, \ldots, n\}^p$ modulo n in $\mathbb{Z}^p$ and $\mathcal{E} = \{\{v, w\} : d(v, w) = 1\}$; in this case d is the L^1 norm

$$d(v, w) = \sum_{i=1}^{p} |v_i - w_i|.$$

Considering the case where we call vertex v a local extreme value if the value C_v exceeds the values C_w over the immediate neighbors w of v, we take

$$\mathcal{V}_v = \mathcal{V}_v(1) \quad \text{and that} \quad |\mathcal{V}_v(1)| = 1 + 2p,$$

the 1 accounting for v itself, and then $2p$ for the number of neighbors at distance 1 from v, which differ from v by either $+1$ or -1 in exactly one coordinate.

Lemma 4.1, (21), and $|X_v| \le 1$ yield

$$|Y^s - Y| \le \max_v |\mathcal{V}_v(2)| = 1 + 2p + \left(2p + 4\binom{p}{2}\right) = 2p^2 + 2p + 1, \tag{22}$$

where the 1 counts v itself, the $2p$ again are the neighbors at distance 1, and the term in the parenthesis accounting for the neighbors at distance 2, $2p$ of them differing in exactly one coordinate by $+2$ or -2, and $4\binom{p}{2}$ of them differing by either $+1$ or -1 in exactly two coordinates. Note that we have used the assumption $n \ge 5$ here, and continue to do so below.

Now letting C_v have a continuous distribution, without loss of generality we can assume $C_v \sim \mathcal{U}[0, 1]$. As any vertex has chance $1/|\mathcal{V}_v|$ of having the largest value in its neighborhood, for the mean μ of Y we have

$$\mu = \frac{n}{2p + 1}. \tag{23}$$

To begin the calculation of the variance, note that when v and w are neighbors they cannot both be maxima, so $X_v X_w = 0$ and therefore, for $d(v, w) = 1$,

$$\mathrm{Cov}(X_v, X_w) = -(EX_v)^2 = -\frac{1}{(2p + 1)^2}.$$

If the distance between v and w is 3 or more, X_v and X_w are functions of disjoint sets of independent variables, and hence are independent.

When $d(w, v) = 2$ there are two cases, as v and w may have either 1 or 2 neighbors in common, and

$$\begin{aligned} &EX_v X_w = \\ &\quad P(U > U_j, V > V_j, j = 1, \ldots, m - k \quad \text{and} \quad U > U_j, V > U_j, j = m - k + 1, \ldots, m), \end{aligned}$$

where m is the number of vertices over which v and w are extreme, so $m = 2p$, and $k = 1$ and $k = 2$ for the number of neighbors in common. For $k = 1, 2, \ldots$, letting $M_k = \max\{U_{m-k+1}, \ldots, U_m\}$, as the variables X_v and X_w are conditionally independent given $U_{m-k+1}, \ldots, U_m$

$$\begin{aligned} E(X_v X_w | U_{m-k+1}, \ldots, U_m) &= P(U > U_j, j = 1, \ldots, m | U_{m-k+1}, \ldots, U_m)^2 \\ &= \frac{1}{(m - k + 1)^2}(1 - M_k^{m-k+1})^2, \end{aligned} \tag{24}$$

as

$$\begin{aligned} P(U > U_j, j = 1, \ldots, m | U_{m-k+1}, \ldots, U_m) &= \int_{M_k}^1 \int_0^u \cdots \int_0^u du_1 \cdots du_{m-k} du \\ &= \int_{M_k}^1 u^{m-k} du \\ &= \frac{1}{m-k+1}(1 - M_k^{m-k+1}). \end{aligned}$$

Since $P(M_k \le x) = x^k$ on $[0,1]$, we have

$$\begin{aligned} EM_k^{m-k+1} &= k \int_0^1 x^{m-k+1} x^{k-1} dx = \frac{k}{m+1} \quad \text{and} \\ E(M_k^{m-k+1})^2 &= k \int_0^1 x^{2(m-k+1)} x^{k-1} dx = \frac{k}{2m-k+2}. \end{aligned}$$

Hence, averaging (24) over $U_{m-k+1}, \ldots, U_m$ yields

$$EX_v X_w = \frac{2}{(m+1)(2(m+1)-k)}.$$

For $n \ge 3$, when $m = 2p$, for $k = 1$ and 2 we obtain

$$\mathrm{Cov}(X_v, X_w) = \frac{1}{(2p+1)^2(2(2p+1)-1)} \quad \text{and} \quad \mathrm{Cov}(X_v, X_w) = \frac{2}{(2p+1)^2(2(2p+1)-2)}, \quad \text{respectively.}$$

For $n \ge 5$, of the $2p + 4\binom{p}{2}$ vertices w that are at distance 2 from v, $2p$ of them share 1 neighbor in common with v, while the remaining $4\binom{p}{2}$ of them share 2 neighbors. Hence,

$$\begin{aligned} \sigma^2 &= \sum_{v \in V} \mathrm{Var}(X_v) + \sum_{v \ne w} \mathrm{Cov}(X_v, X_w) \\ &= \sum_{v \in V} \mathrm{Var}(X_v) + \sum_{d(v,w)=1} \mathrm{Cov}(X_v, X_w) + \sum_{d(v,w)=2} \mathrm{Cov}(X_v, X_w) \\ &= n \left(\frac{2p}{(2p+1)^2} - 2p \frac{1}{(2p+1)^2} + 2p \frac{1}{(2p+1)^2(2(2p+1)-1)} + 4\binom{p}{2} \frac{2}{(2p+1)^2(2(2p+1)-2)} \right) \\ &= n \frac{2p}{(2p+1)^2} \left(\frac{1}{(2(2p+1)-1)} + \frac{2(p-1)}{(2(2p+1)-2)} \right) \\ &= n \left(\frac{4p^2 - p - 1}{(2p+1)^2(4p+1)} \right). \end{aligned} \tag{25}$$

We conclude that (2) of Theorem 1.1 holds with $A = C\mu/\sigma^2$ and $B = C/2\sigma$ with μ, σ^2 and C given by (23), (25) and (22), respectively, that is,

$$A = \frac{(2p+1)(4p+1)(2p^2+2p+1)}{4p^2-p-1} \quad \text{and} \quad B = \frac{2p^2+2p+1}{2\sqrt{n\left(\frac{4p^2-p-1}{(2p+1)^2(4p+1)}\right)}}.$$

4.3 Urn allocation

In the classical urn allocation model n balls are thrown independently into one of m urns, where, for $i = 1, \ldots, m$, the probability a ball lands in the i^{th} urn is p_i, with $\sum_{i=1}^m p_i = 1$. A much studied quantity of

interest is the number of nonempty urns, for which Kolmogorov distance bounds to the normal were obtained in [11] and [27]. In [11], bounds were obtained for the uniform case where $p_i = 1/m$ for all $i = 1, \ldots, m$, while the bounds in [27] hold for the nonuniform case as well. In [25] the author considers the normal approximation for the number of isolated balls, that is, the number of urns containing exactly one ball, and obtains Kolmogorov distance bounds to the normal. Using the coupling provided in [25], we derive right tail inequalities for the number of non-isolated balls, or, equivalently, left tail inequalities for the number of isolated balls.

For $i = 1, \ldots, n$ let X_i denote the location of ball i, that is, the number of the urn into which ball i lands. The number Y of non-isolated balls is given by

$$Y = \sum_{i=1}^{n} 1(M_i > 0) \quad \text{where} \quad M_i = -1 + \sum_{j=1}^{n} 1(X_j = X_i).$$

We first consider the uniform case. A construction in [25] produces a coupling of Y to Y^s, having the Y size biased distribution, which satisfies $|Y^s - Y| \leq 2$. Given a realization of $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$, the coupling proceeds by first selecting a ball I, uniformly from $\{1, 2, \ldots, n\}$, and independently of $\mathbf{X}$. Depending on the outcome of a Bernoulli variable $\mathcal{B}$, whose distribution depends on the number of balls found in the urn containing I, a different ball J will be imported into the urn that contains ball I. In some additional detail, let $\mathcal{B}$ be a Bernoulli variable with success probability $P(\mathcal{B} = 1) = \pi_{M_I}$, where

$$\pi_k = \begin{cases} \frac{P(N>k|N>0)-P(N>k)}{P(N=k)(1-k/(n-1))} & \text{if } 0 \leq k \leq n-2 \\ 0 & \text{if } k = n-1, \end{cases}$$

with $N \sim \text{Bin}(1/m, n-1)$. Now let J be uniformly chosen from $\{1, 2, \ldots, n\} \setminus \{I\}$, independent of all other variables. Lastly, if $\mathcal{B} = 1$, move ball J into the same urn as I. It is clear that $|Y' - Y| \leq 2$, as at most the occupancy of two urns can affected by the movement of a single ball. We also note that if $M_I = 0$, which happens when ball I is isolated, $\pi_0 = 1$, so that I becomes no longer isolated after relocating ball J. We refer the reader to [25] for a full proof that this procedure produces a coupling of Y to a variable with the Y size biased distribution.

For the uniform case, the following explicit formulas for μ and σ^2 can be found in Theorem II.1.1 of [18],

$$\begin{aligned} \mu &= n\left(1 - \left(1 - \frac{1}{m}\right)^{n-1}\right) \quad \text{and} \\ \sigma^2 &= (n-\mu) + \frac{(m-1)n(n-1)}{m}\left(1 - \frac{2}{m}\right)^{n-2} - (n-\mu)^2 \\ &= n\left(1 - \frac{1}{m}\right)^{n-1} + \frac{(m-1)n(n-1)}{m}\left(1 - \frac{2}{m}\right)^{n-2} - n^2\left(1 - \frac{1}{m}\right)^{2n-2}. \end{aligned} \tag{26}$$

Hence with μ and σ^2 as in (26), we can apply (3) of Theorem 1.1 for Y, the number of non isolated balls with $C = 2$, $A = 2\mu/\sigma^2$ and $B = 1/\sigma$.

Taking limits in (26), if m and n both go to infinity in such a way that $n/m \to \alpha \in (0, \infty)$, the mean μ and variance σ^2 obey

$$\mu \asymp n(1 - e^{-\alpha}) \quad \text{and } \sigma^2 \asymp n g(\alpha)^2 \text{ where } g(\alpha)^2 = e^{-\alpha} - e^{-2\alpha}(\alpha^2 - \alpha + 1) > 0 \text{ for all } \alpha \in (0, \infty),$$

where for positive functions f and h depending on n we write $f \asymp h$ when $\lim_{n\to\infty} f/h = 1$.

Hence, in this limiting case A and B satisfy

$$A \asymp \frac{2(1 - e^{-\alpha})}{e^{-\alpha} - e^{-2\alpha}(\alpha^2 - \alpha + 1)} \quad \text{and} \quad B \asymp \frac{1}{\sqrt{n} g(\alpha)}.$$

In the nonuniform case similar results hold with some additional conditions. Letting

$$||p|| = \sup_{1\le i\le m} p_i \quad \text{and} \quad \gamma = \gamma(n) = \max(n||p||, 1),$$

in [25] it is shown that when $||p|| \le 1/11$ and $n \ge 83\gamma^2(1+3\gamma+3\gamma^2)e^{1.05\gamma}$, there exists a coupling such that

$$|Y^s - Y| \le 3 \quad \text{and} \quad \frac{\mu}{\sigma^2} \le 8165\gamma^2 e^{2.1\gamma}.$$

Now also using Theorem 2.4 in [25] for a bound on σ^2, we find that (3) of Theorem 1.1 holds with

$$A = 24,495\,\gamma^2 e^{2.1\gamma} \quad \text{and} \quad B = \frac{1.5\sqrt{7776}\,\gamma e^{1.05\gamma}}{n\sqrt{\sum_{i=1}^m p_i^2}}.$$

4.4 An application to coverage processes

We consider the following coverage process, and associated coupling, from [14]. Given a collection $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ of independent, uniformly distributed points in the d dimensional torus of volume n, that is, the cube $C_n = [0, n^{1/d})^d \subset \mathbb{R}^d$ with periodic boundary conditions, let V denote the total volume of the union of the n balls of fixed radius ρ centered at these n points, and S the number of balls isolated at distance ρ, that is, those points for which none of the other $n-1$ points lie within distance ρ. The random variables V and S are of fundamental interest in stochastic geometry, see [17] and [24]. If $n \to \infty$ and ρ remains fixed, both V and S satisfy a central limit theorem [17, 22, 26]. The L^1 distance of V, properly standardized, to the normal is studied in [9] using Stein's method. The quality of the normal approximation to the distributions of both V and S, in the Kolmogorov metric, is studied in [14] using Stein's method via size bias couplings.

In more detail, for $x \in C_n$ and $r > 0$ let $B_r(x)$ denote the ball of radius r centered at x, and $B_{i,r} = B(U_i, r)$. The covered volume V and number of isolated balls S are given, respectively, by

$$V = \text{Volume}(\bigcup_{i=1}^n B_{i,\rho}) \quad \text{and} \quad S = \sum_{i=1}^n \mathbf{1}\{(\mathcal{U}_n \cap B_{i,\rho} = \{U_i\}\}. \tag{27}$$

We will derive concentration of measure inequalities for V and S with the help of the bounded size biased couplings in [14].

Assume $d \ge 1$ and $n \ge 4$. Denote the mean and variance of V by μ_V and σ_V^2, respectively, and likewise for S, leaving their dependence on n and ρ implicit. Let $\pi_d = \pi^{d/2}/\Gamma(1+d/2)$, the volume of the unit sphere in $\mathbb{R}^d$, and for fixed ρ let $\phi = \pi_d \rho^d$. For $0 \le r \le 2$ let $\omega_d(r)$ denote the volume of the union of two unit balls with centers r units apart. We have $\omega_1(r) = 2 + r$, and

$$\omega_d(r) = \pi_d + \pi_{d-1}\int_0^r (1 - (t/2)^2)^{(d-1)/2} dt, \quad \text{for } d \ge 2.$$

From [14], the means of V and S are given by

$$\mu_V = n\,(1 - (1-\phi/n)^n) \quad \text{and} \quad \mu_S = n(1-\phi/n)^{n-1}, \tag{28}$$

and their variances by

$$\sigma_V^2 = n\int_{B_{2\rho}(\mathbf{0})} \left(1 - \frac{\rho^d \omega_d(|y|/\rho)}{n}\right)^n dy + n(n - 2^d\phi)\left(1 - \frac{2\phi}{n}\right)^n - n^2(1-\phi/n)^{2n}, \tag{29}$$

and

$$\begin{aligned}\sigma_S^2 &= n(1-\phi/n)^{n-1}(1 - (1-\phi/n)^{n-1}) \\ &\quad + (n-1)\int_{B_{2\rho}(\mathbf{0})\setminus B_\rho(\mathbf{0})} \left(1 - \frac{\rho^d\omega_d(|y|/\rho)}{n}\right)^{n-2} dy \\ &\quad + n(n-1)\left(\left(1 - \frac{2^d\phi}{n}\right)\left(1 - \frac{2\phi}{n}\right)^{n-2} - \left(1 - \frac{\phi}{n}\right)^{2n-2}\right).\end{aligned} \tag{30}$$

It is shown in [14], by using a coupling similar to the one briefly described for the urn allocation problem in Section 4.3, that one can construct V^s with the V size bias distribution which satisfies $|V^s - V| \leq \phi$. Hence (2) of Theorem 1.1 holds for V with

$$A_V = \frac{\phi \mu_V}{\sigma_V^2} \quad \text{and} \quad B_V = \frac{\phi}{2\sigma_V},$$

where μ_V and σ_V^2 are given in (28) and (29), respectively. Similarly, with $Y = n - S$ the number of non-isolated balls, it is shown that Y^s with Y size bias distribution can be constructed so that $|Y^s - Y| \leq \kappa_d + 1$, where κ_d denotes the maximum number of open unit balls in d dimensions that can be packed so they all intersect an open unit ball in the origin, but are disjoint from each other. Hence (2) of Theorem 1.1 holds for Y with

$$A_Y = \frac{(\kappa_d + 1)(n - \mu_S)}{\sigma_S^2} \quad \text{and} \quad B_Y = \frac{\kappa_d + 1}{2\sigma_S}.$$

To see how the A_V, A_Y and B_V, B_Y behave as $n \to \infty$, let

$$J_{r,d}(\rho) = d\pi_d \int_0^r \exp(-\rho^d \omega_d(t)) t^{d-1} dt,$$

and define

$$\begin{aligned} g_V(\rho) &= \rho^d J_{2,d}(\rho) - (2^d \phi + \phi^2) e^{-2\phi} \quad \text{and} \\ g_S(\rho) &= e^{-\phi} - (1 + (2^d - 2)\phi + \phi^2) e^{-2\phi} + \rho^d (J_{2,d}(\rho) - J_{1,d}(\rho)). \end{aligned}$$

Then, again from [14],

$$\begin{aligned} \lim_{n\to\infty} n^{-1} \mu_V = \lim_{n\to\infty} (1 - n^{-1} \mu_S) &= 1 - e^{-\phi}, \\ \lim_{n\to\infty} n^{-1} \sigma_V^2 &= g_V(\rho) > 0, \quad \text{and} \\ \lim_{n\to\infty} n^{-1} \sigma_S^2 &= g_S(\rho) > 0. \end{aligned}$$

Hence, B_V and B_Y tend to zero at rate $n^{-1/2}$, and

$$\lim_{n\to\infty} A_V = \frac{\phi(1 - e^{-\phi})}{g_V(\rho)}, \quad \text{and} \quad \lim_{n\to\infty} A_Y = \frac{(\kappa_d + 1)(1 - e^{-\phi})}{g_S(\rho)}.$$

4.5 The lightbulb problem

The following stochastic process, known informally as the 'lightbulb process', arises in a pharmaceutical study of dermal patches, see [29]. Changing dermal receptors to lightbulbs allows for a more colorful description. Consider n lightbulbs, each operated by a switch. At day zero, none of the bulbs are on. At day r for $r = 1, \ldots, n$, the position of r of the n switches are selected uniformly to be changed, independent of the past. One is interested in studying the distribution of the number of lightbulbs which are switched on at the terminal time n. The process just described is Markovian, and is studied in some detail in [34]. In [16] the authors use Stein's method to derive a bound to the normal via a monotone, bounded size bias coupling. Borrowing this coupling here allows for the application of Theorem 1.1 to obtain concentration of measure inequalities for the lightbulb problem. We begin with a more detailed description of the process.

For $r = 1, \ldots, n$, let $\{X_{rk}, k = 1, \ldots, n\}$ have distribution

$$P(X_{r1} = e_1, \ldots, X_{rn} = e_n) = \binom{n}{r}^{-1} \quad \text{for all } e_k \in \{0, 1\} \text{ with } \textstyle\sum_{k=1}^n e_k = r,$$

and let these collections of variables be independent over r. These 'switch variables' X_{rk} indicate whether or not on day r bulb k had its status changed. With

$$Y_k = \left(\sum_{r=1}^{n} X_{rk} \right) \bmod 2$$

therefore indicating the status of bulb k at time n, the number of bulbs switched on at the terminal time is

$$Y = \sum_{k=1}^{n} Y_k.$$

From [29], the mean μ and variance σ^2 of Y are given by

$$\mu = \frac{n}{2} \left(1 - \prod_{i=1}^{n} \left(1 - \frac{2i}{n} \right) \right), \tag{31}$$

and

$$\sigma^2 = \frac{n}{4} \left[1 - \prod_{i=1}^{n} \left(1 - \frac{4i}{n} + \frac{4i(i-1)}{n(n-1)} \right) \right] + \frac{n^2}{4} \left[\prod_{i=1}^{n} \left(1 - \frac{4i}{n} + \frac{4i(i-1)}{n(n-1)} \right) - \prod_{i=1}^{n} \left(1 - \frac{2i}{n} \right)^2 \right]. \tag{32}$$

Note that when n is even $\mu = n/2$ exactly, as the product in (31) is zero, containing the term $i = n/2$. By results in [29], in the odd case $\mu = (n/2)(1 + O(e^{-n}))$, and in both the even and odd cases $\sigma^2 = (n/4)(1 + O(e^{-n}))$.

The following construction, given in [16] for the case where n is even, couples Y to a variable Y^s having the Y size bias distribution such that

$$Y \le Y^s \le Y + 2, \tag{33}$$

that is, the coupling is monotone, with difference bounded by 2. For every $i \in \{1, \ldots, n\}$ construct the collection of variables $\mathbf{Y}^i$ from $\mathbf{Y}$ as follows. If $Y_i = 1$, that is, if bulb i is on, let $\mathbf{Y}^i = \mathbf{Y}$. Otherwise, with $J^i = \mathcal{U}\{j : Y_{n/2,j} = 1 - Y_{n/2,i}\}$, let $\mathbf{Y}^i = \{Y^i_{rk} : r, k = 1, \ldots, n\}$ where

$$Y^i_{rk} = \begin{cases} Y_{rk} & r \neq n/2 \\ Y_{n/2,k} & r = n/2, k \notin \{i, J^i\} \\ Y_{n/2,J^i} & r = n/2, k = i \\ Y_{n/2,i} & r = n/2, k = J^i, \end{cases}$$

and let $Y^i = \sum_{k=1}^{n} Y^i_k$ where

$$Y^i_k = \left(\sum_{r=1}^{n} Y^i_{rk} \right) \bmod 2.$$

Then, with I uniformly chosen from $\{1, \ldots, n\}$ and independent of all other variables, it is shown in [16] that the mixture $Y^s = Y^I$ has the Y size biased distribution, essentially due to the fact that

$$\mathcal{L}(\mathbf{Y}^i) = \mathcal{L}(\mathbf{Y} | Y_i = 1) \quad \text{for all } i = 1, \ldots, n.$$

It is not difficult to see that Y^s satisfies (33). If $Y_I = 1$ then $\mathbf{X}^I = \mathbf{X}$, and so in this case $Y^s = Y$. Otherwise $Y_I = 0$, and for the given I the collection $\mathbf{Y}^I$ is constructed from $\mathbf{Y}$ by interchanging the stage $n/2$, unequal, switch variables $Y_{n/2,I}$ and $Y_{n/2,J^I}$. If $Y_{J^I} = 1$ then after the interchange $Y'_I = 1$ and $Y'_{J^I} = 0$, in which case $Y^s = Y$. If $Y_{J^I} = 0$ then after the interchange $Y^I_I = 1$ and $Y^I_{J^I} = 1$, yielding $Y^s = Y + 2$. We conclude that for the case n even $C = 2$ and (2) and (3) of Theorem 1.1 hold with

$$A = n/\sigma^2 \quad \text{and} \quad B = 1/\sigma \tag{34}$$

where σ^2 is given by (32).

For the coupling in the odd case, $n = 2m+1$ say, due to the parity issue, [16] considers a random variable V close to Y constructed as follows. In all stages but stage m and $m+1$ let the switch variables which will yield V be the same as those for Y. In stage m, however, with probability $1/2$ one applies an additional switch variable, and in stage $m+1$, with probability $1/2$, one switch variable fewer. In this way the switch variables in these two stages have the same, symmetric distribution and are close to the switch variables for Y. In particular, as at most two switch variables are different in the configuration for V, we have $|V-Y| \leq 2$. Helped by the symmetry, one may couple V to a variable V^s with the V size bias distribution as in the even case, obtaining $V \leq V^s \leq V+2$. Hence (2) and (3) of Theorem 1.1 hold for V as for the even case with values given in (34), where $\mu = n/2$ and $\sigma^2 = (n/4)(1+O(e^{-n})$. Since $|V-Y| \leq 2$, by replacing t by $t+2/\sigma$ in the bounds for V one obtains bounds for the odd case Y.

5 Applications: unbounded couplings

One of the major drawbacks of Theorem 1.1 is the hypothesis that $|Y^s - Y|$ be almost surely bounded with probability one. In this section we derive concentration of measure inequalities for two examples where $Y^s - Y$ is not bounded: the number of isolated vertices in the Erdös-Rényi random graph model, and the nonnegative infinitely divisible distributions with certain associated moment generating functions which satisfy a boundedness condition. For the latter, compound Poisson distributions will be our main illustration.

5.1 Number of isolated vertices in the Erdös Rényi random graph model

Let $K_{n,p}$ be the random graph on the vertices $\mathcal{V} = \{1, 2, \ldots, n\}$, with the indicators X_{vw} of the presence of edges between two unequal vertices v and w being independent Bernoulli $p \in (0,1)$ variables, and $X_{vv} = 0$ for all $v \in \mathcal{V}$. Recall that the degree of a vertex $v \in \mathcal{V}$ is the number of edges incident on v,

$$d(v) = \sum_{w \in \mathcal{V}} X_{vw}. \tag{35}$$

The problem of approximating the distribution of the number of vertices v with degree $d(v) = d$ for some fixed d was considered in [5], and a smooth function bound to the multivariate normal for a vector whose components count the number of vertices of some fixed degrees was given in [15].

Here we study the number of isolated vertices $Y_{n,p}$ of $K_{n,p}$, that is, those vertices which have no incident edges, given by

$$Y_{n,p} = \sum_{v \in \mathcal{V}} 1(d(v) = 0).$$

In [19], the mean μ and variance σ^2 of $Y_{n,p}$ are given as

$$\mu_{n,p} = n(1-p)^{n-1} \quad \text{and} \quad \sigma^2_{n,p} = n(1-p)^{n-1}(1 + np(1-p)^{n-2} - (1-p)^{n-2}), \tag{36}$$

where also Kolmogorov distance bounds to the normal were obtained, and asymptotic normality shown when

$$n^2 p \to \infty \quad \text{and} \quad np - \log(n) \to -\infty.$$

O'Connell [23] shows an asymptotic large deviation principle holds for $Y_{n,p}$. Raič [28] obtained nonuniform large deviation bounds in some generality for random variables W with $E(W) = 0$ and $\mathrm{Var}(W) = 1$ of the form,

$$\frac{P(W \geq t)}{1 - \Phi(t)} \leq e^{t^3 \beta(t)/6}(1 + Q(t)\beta(t)) \quad \text{for all } t \geq 0, \tag{37}$$

where $\Phi(t)$ denotes the distribution function of a standard normal variate and $Q(t)$ is a quadratic in t. Although in general the expression for $\beta(t)$ is not simple, when W is $Y_{n,p}$ properly standardized and $np \to c$ as $n \to \infty$, then (37) holds for all n sufficiently large with

$$\beta(t) = \frac{C_1}{\sqrt{n}} \exp\left(\frac{C_2 t}{\sqrt{n}} + C_3 (e^{C_4 t/\sqrt{n}} - 1) \right)$$

for some constants C_1, C_2, C_3 and C_4. For t of order $n^{1/2}$, for instance, the function $\beta(t)$ will be small as $n \to \infty$, allowing an approximation of the deviation probability $P(W \geq t)$ by the normal, to within some factors. Theorem 5.1 below, by contrast, provides a non-asymptotic bound, that is, not relying on any limiting relations between n and p, with explicit constants, which hold for every n. Moreover, the bound is of order e^{-at^2} over some range of t, and of worst case order e^{-bt}, for the right tail by (40), and e^{-ct^2} by (39) for the left tail, where a, b and c are explicit, with the bounds holding for all $t \in \mathbb{R}$.

For notational ease, we keep the dependence on n and p implicit in the sequel.

Theorem 5.1. *Let K denote the random graph on n vertices where each edge is present with probability $p \in (0,1)$, independently of all other edges, and let Y denote the number of isolated vertices in K. Then for all $t > 0$,*

$$P\left(\frac{Y - \mu}{\sigma} \geq t \right) \leq \inf_{\theta \geq 0} \exp(-\theta t + H(\theta)) \quad \textit{where} \quad H(\theta) = \frac{\mu}{2\sigma^2} \int_0^\theta s \gamma_s ds, \tag{38}$$

with the mean μ and variance σ^2 of Y given in (36), and

$$\gamma_s = 2e^{2s} \left(1 + \frac{pe^s}{1-p} \right)^n + \beta + 1 \quad \textit{where } \beta = (1-p)^{-n}.$$

For the left tail, for all $t > 0$,

$$P\left(\frac{Y - \mu}{\sigma} \leq -t \right) \quad \leq \quad \exp\left(-\frac{t^2}{2} \frac{\sigma^2}{\mu(\beta+1)} \right). \tag{39}$$

Remark 5.1. *Though the minimization in (38) is admittedly cumbersome, useful bounds may be obtained by restricting the minimization to $\theta \in [0, \theta_0]$ for some θ_0. In this case, as γ_s is an increasing function of s, we have*

$$H(\theta) \leq \frac{\mu}{4\sigma^2} \gamma_{\theta_0} \theta^2 \quad \textit{for } \theta \in [0, \theta_0].$$

The quadratic $-\theta t + \mu \gamma_{\theta_0} \theta^2/(4\sigma^2)$ in θ is minimized at $\theta = 2t\sigma^2/(\mu\gamma_{\theta_0})$. When this value falls in $[0, \theta_0]$ we obtain the first bound in (40), while otherwise setting $\theta = \theta_0$ yields the second.

$$P(\frac{Y - \mu}{\sigma} \geq t) \leq \begin{cases} \exp(-\frac{t^2\sigma^2}{\mu\gamma_{\theta_0}}) & \textit{for } t \in [0, \theta_0 \mu \gamma_{\theta_0}/(2\sigma^2)] \\ \exp(-\theta_0 t + \frac{\mu\gamma_{\theta_0}\theta_0^2}{4\sigma^2}) & \textit{for } t \in (\theta_0 \mu \gamma_{\theta_0}/(2\sigma^2), \infty). \end{cases} \tag{40}$$

Though Theorem 5.1 is not an asymptotic, as it gives bounds for any specific n and p, when $np \to c$ as $n \to \infty$ we have

$$\frac{\sigma^2}{\mu} \to 1 + ce^{-c} - e^{-c}, \quad \beta + 1 \to e^c + 1 \quad \textit{and} \quad \gamma_s \to 2e^{2s + ce^s} + e^c + 1 \quad \textit{as } n \to \infty.$$

Hence, the left tail bound (39), for example, in this asymptotic behaves as

$$\lim_{n \to \infty} \exp\left(-\frac{t^2}{2} \frac{\sigma^2}{\mu(\beta+1)} \right) = \exp\left(-\frac{t^2}{2} \frac{1 + ce^{-c} - e^{-c}}{e^c + 1} \right).$$

Proof. We first review the construction of Y^s, having the Y size bias distribution, as given in [15]. Let K, a particular realization of $K(n,p)$, be given, and let Y be the number of isolated vertices for this realization. To size bias Y, choose one of the n vertices of K uniformly. If the chosen vertex, say V, is already isolated, we do nothing and set $K^s = K$. Otherwise obtain K^s by deleting all the edges connected to K. Then Y^s, the number of isolated vertices of K^s, has the Y size biased distribution.

To derive the needed properties of this coupling, let $N(v)$ be the set of neighbors of $v \in \mathcal{V}$, and $\mathcal{T}$ the collection of isolated vertices of K, that is, with $d(v)$, the degree of v, given in (35),

$$N(v) = \{w : X_{vw} = 1\} \quad \text{and} \quad \mathcal{T} = \{v : d(v) = 0\}.$$

Note that $Y = |\mathcal{T}|$. Since all edges incident to the chosen V are removed in order to form K^s, any neighbor of V which had degree one thus becomes isolated, and V also becomes isolated if it was not so earlier. As all others vertices are otherwise unaffected, as far as their being isolated or not, we have

$$Y^s - Y = d_1(V) + 1(d(V) \neq 0) \quad \text{where} \quad d_1(V) = \sum_{w \in N(V)} 1(d(w) = 1), \tag{41}$$

so in particular the coupling is monotone. Since $d_1(V) \leq d(V)$, (41) yields

$$Y^s - Y \leq d(V) + 1. \tag{42}$$

By (5), using that the coupling is monotone, for $\theta \geq 0$ we have

$$\begin{aligned} E(e^{\theta Y^s} - e^{\theta Y}) &\leq \frac{\theta}{2} E\left((Y^s - Y)(e^{\theta Y^s} + e^{\theta Y})\right) \\ &= \frac{\theta}{2} E\left(\exp(\theta Y)(Y^s - Y)\left(\exp(\theta(Y^s - Y)) + 1\right)\right) \\ &= \frac{\theta}{2} E\left\{\exp(\theta Y) E((Y^s - Y)(\exp(\theta(Y^s - Y)) + 1)|\mathcal{T})\right\}. \end{aligned} \tag{43}$$

Now using that $Y^s = Y$ when $V \in \mathcal{T}$, and (42), we have

$$\begin{aligned} &E((Y^s - Y)(\exp(\theta(Y^s - Y)) + 1)|\mathcal{T}) \\ &\quad \leq E((d(V) + 1)(\exp(\theta(d(V) + 1)) + 1)1(V \notin \mathcal{T})|\mathcal{T}) \\ &\quad \leq e^{\theta} E\left(\left(d(V)e^{\theta d(V)} + e^{\theta d(V)} + d(V)\right)1(V \notin \mathcal{T})|\mathcal{T}\right) + 1. \end{aligned} \tag{44}$$

Note that since V is chosen independently of K,

$$\mathcal{L}(d(V)1(V \notin \mathcal{T})|\mathcal{T}) = P(V \notin \mathcal{T})\mathcal{L}(\,Bin(n - 1 - Y, p)|\,Bin(n - 1 - Y, p) > 0) + P(V \in \mathcal{T})\delta_0, \tag{45}$$

where δ_0 is point mass at zero. By (45), and that the mass function of the conditioned binomial there is

$$P(d(V) = k|\mathcal{T}, V \notin \mathcal{T}) = \begin{cases} \binom{n-1-Y}{k} \frac{p^k (1-p)^{n-1-Y-k}}{1-(1-p)^{n-1-Y}} & \text{for } 1 \leq k \leq n - 1 - Y \\ 0 & \text{otherwise,} \end{cases}$$

it can be easily verified that the conditional moment generating function of $d(V)$ and its first derivative are bounded by

$$\begin{aligned} E(e^{\theta d(V)} 1(V \notin \mathcal{T})|\mathcal{T}) &\leq \frac{(pe^{\theta} + 1 - p)^{n-1-Y} - (1-p)^{n-1-Y}}{1 - (1-p)^{n-1-Y}} \quad \text{and} \\ E(d(V)e^{\theta d(V)} 1(V \notin \mathcal{T})|\mathcal{T}) &\leq \frac{(n - 1 - Y)(pe^{\theta} + 1 - p)^{n-2-Y} pe^{\theta}}{1 - (1-p)^{n-1-Y}}. \end{aligned}$$

By the mean value theorem applied to the function $f(x) = x^{n-1-Y}$, for some $\xi \in (1-p, 1)$ we have

$$1-(1-p)^{n-1-Y} = f(1) - f(1-p) = (n-1-Y)p\xi^{n-2-Y} \geq (n-1-Y)p(1-p)^n.$$

Hence, recalling $\theta \geq 0$,

$$\begin{aligned} E(d(V)e^{\theta d(V)}1(V \notin \mathcal{T})|\mathcal{T}) &\leq \frac{(n-1-Y)(pe^\theta + 1 - p)^n pe^\theta}{1-(1-p)^{n-1-Y}} \\ &\leq \frac{(n-1-Y)(pe^\theta + 1 - p)^n pe^\theta}{(n-1-Y)p(1-p)^n} \\ &= \alpha_\theta \quad \text{where} \quad \alpha_\theta = e^\theta \left(1 + \frac{pe^\theta}{1-p}\right)^n. \end{aligned} \tag{46}$$

Similarly applying the mean value theorem to $f(x) = (x+1-p)^{n-1-Y}$, for some $\xi \in (0, pe^\theta)$ we have

$$\begin{aligned} E(e^{\theta d(V)}1(V \notin \mathcal{T})|\mathcal{T}) &\leq \frac{(n-1-Y)(\xi + (1-p))^{n-2-Y} pe^\theta}{1-(1-p)^{n-1-Y}} \\ &\leq \frac{(n-1-Y)(pe^\theta + (1-p))^{n-2-Y} pe^\theta}{1-(1-p)^{n-1-Y}} \\ &\leq \alpha_\theta, \end{aligned} \tag{47}$$

as in (46).

Next, to handle the second to last term in (44) consider

$$E(d(V)1(V \notin \mathcal{T})|\mathcal{T}) \leq \frac{(n-1-Y)p}{1-(1-p)^{n-1-Y}} \leq \frac{(n-1-Y)p}{(n-1-Y)p(1-p)^n} = \beta \quad \text{where} \quad \beta = (1-p)^{-n}. \tag{48}$$

Applying inequalities (46),(47) and (48) to (44) yields

$$E((Y^s - Y)(\exp(\theta(Y^s - Y)) + 1)|\mathcal{T}) \leq \gamma_\theta \quad \text{where} \quad \gamma_\theta = 2e^\theta \alpha_\theta + \beta + 1. \tag{49}$$

Hence we obtain, using (43),

$$E(e^{\theta Y^s} - e^{\theta Y}) \leq \frac{\theta\gamma_\theta}{2} E(e^{\theta Y}) \quad \text{for all } \theta \geq 0.$$

Letting $m(\theta) = E(e^{\theta Y})$ thus yields

$$m'(\theta) = E(Ye^{\theta Y}) = \mu E(e^{\theta Y^s}) \leq \mu \left(1 + \frac{\theta\gamma_\theta}{2}\right) m(\theta). \tag{50}$$

Setting

$$M(\theta) = E(\exp(\theta(Y-\mu)/\sigma)) = e^{-\theta\mu/\sigma} m(\theta/\sigma),$$

differentiating and using (50), we obtain

$$\begin{aligned} M'(\theta) &= \frac{1}{\sigma} e^{-\theta\mu/\sigma} m'(\theta/\sigma) - \frac{\mu}{\sigma} e^{-\theta\mu/\sigma} m(\theta/\sigma) \\ &\leq \frac{\mu}{\sigma} e^{-\theta\mu/\sigma} (1 + \frac{\theta\gamma_\theta}{2\sigma}) m(\theta/\sigma) - \frac{\mu}{\sigma} e^{-\theta\mu/\sigma} m(\theta/\sigma) \\ &= e^{-\theta\mu/\sigma} \frac{\mu\theta\gamma_\theta}{2\sigma^2} m(\theta/\sigma) = \frac{\mu\theta\gamma_\theta}{2\sigma^2} M(\theta). \end{aligned} \tag{51}$$

Since $M(0) = 1$, (51) yields upon integration of $M'(s)/M(s)$ over $[0, \theta]$,

$$\log(M(\theta)) \leq H(\theta) \quad \text{so that} \quad M(\theta) \leq \exp(H(\theta)) \quad \text{where} \quad H(\theta) = \frac{\mu}{2\sigma^2} \int_0^\theta s\gamma_s ds.$$

Hence for $t \geq 0$,

$$P(\frac{Y-\mu}{\sigma} \geq t) \quad \leq \quad P(\exp(\frac{\theta(Y-\mu)}{\sigma}) \geq e^{\theta t}) \leq e^{-\theta t} M(\theta) \leq \exp(-\theta t + H(\theta)).$$

As the inequality holds for all $\theta \geq 0$, it holds for the θ achieving the minimal value, proving (38).

For the left tail bound let $\theta < 0$. Since $Y^s \geq Y$ and $\theta < 0$, using (5) and (42) we obtain

$$\begin{aligned} E(e^{\theta Y} - e^{\theta Y^s}) &\leq \frac{|\theta|}{2} E\left((e^{\theta Y} + e^{\theta Y^s})(Y^s - Y)\right) \\ &\leq |\theta| E(e^{\theta Y}(Y^s - Y)) \\ &= |\theta| E(e^{\theta Y} E(Y^s - Y|\mathcal{T})) \\ &\leq |\theta| E(e^{\theta Y} E((d(V)+1)1(V \notin \mathcal{T}))|\mathcal{T})). \end{aligned}$$

Applying (48) we obtain

$$E(e^{\theta Y} - e^{\theta Y^s}) \leq (\beta+1)|\theta| E(e^{\theta Y}),$$

and therefore

$$m'(\theta) = \mu E(e^{\theta Y^s}) \geq \mu\,(1 + (\beta+1)\theta)\, m(\theta).$$

Hence for $\theta < 0$,

$$\begin{aligned} M'(\theta) &= \frac{1}{\sigma} e^{-\theta\mu/\sigma} m'(\theta/\sigma) - \frac{\mu}{\sigma} e^{-\theta\mu/\sigma} m(\theta/\sigma) \\ &\geq \frac{\mu}{\sigma} e^{-\theta\mu/\sigma}((1 + (\beta+1)\theta/\sigma) m(\theta/\sigma)) - \frac{\mu}{\sigma} e^{-\theta\mu/\sigma} m(\theta/\sigma) \\ &= \frac{\mu(\beta+1)\theta}{\sigma^2} M(\theta). \end{aligned}$$

Dividing by $M(\theta)$ and integrating over $[\theta, 0]$ yields

$$\log(M(\theta)) \leq \frac{\mu(\beta+1)\theta^2}{2\sigma^2}. \tag{52}$$

The inequality in (52) implies that for all $t > 0$ and $\theta < 0$,

$$P(\frac{Y-\mu}{\sigma} \leq -t) \leq \exp(\theta t + \frac{\mu(\beta+1)\theta^2}{2\sigma^2}).$$

Taking $\theta = -t\sigma^2/(\mu(\beta+1))$ we obtain (39). □ □

5.2 Infinitely divisible and compound Poisson distributions

The examples in this section generalize the application of Theorem 1.1 from the case where Y is Poisson with parameter $\lambda > 0$. In this case, Y admits a bounded coupling to a variable with its size bias distribution due to the characterization

$$E[Yf(Y)] = \lambda E[f(Y+1)] \quad \text{if and only if } Y \sim \text{Poisson}(\lambda), \tag{53}$$

which forms the basis of the Chen-Stein Poisson approximation method, see [8, 4]. In particular we may take $Y^s = Y + 1$, and, therefore $C = 1$. As the mean and variance for the Poisson are equal, and the coupling is monotone, applying Theorem 1.1 we obtain the following result.

Proposition 5.1. *If $Y \sim$ Poisson(λ), then for all $t > 0$,*

$$P\left(\frac{Y-\lambda}{\sqrt{\lambda}} \leq -t\right) \leq \exp\left(-\frac{t^2}{2}\right) \quad \textit{and} \quad P\left(\frac{Y-\lambda}{\sqrt{\lambda}} \geq t\right) \leq \exp\left(-\frac{t^2}{2 + t\lambda^{-1/2}}\right).$$

The Poisson distribution is infinitely divisible, and also a special case of the compound Poisson distributions. We generalize Proposition 5.1 in these directions.

5.2.1 Infinitely divisible distributions

When Y is Poisson then by (53) $Y^s = Y + 1$ and we may write

$$Y^s = Y + X \tag{54}$$

with X and Y independent. Theorem 5.3 of [33] shows that if Y is nonnegative with finite mean then (54) holds if and only if Y is infinitely divisible. Hence, in this case, a coupling of Y to Y^s may be achieved by generating the independent variable X and adding it to Y. Since Y^s is always stochastically larger than Y we must have $X \geq 0$, and therefore this coupling is monotone. In addition $Y^s - Y = X$ so the coupling is bounded if and only if X is bounded. When X is unbounded, Theorem 5.2 provides concentration of measure inequalities for Y under appropriate growth conditions on two generating functions in Y and X. We assume without further mention that Y is nontrivial, and note that therefore the means of both Y and X are positive.

Theorem 5.2. *Let Y have a nonnegative infinitely divisible distribution and suppose that there exists $\gamma > 0$ so that $E(e^{\gamma Y}) < \infty$. Let X have the distribution such that (54) holds when Y and X are independent, and assume $E(Xe^{\gamma X}) = C < \infty$. Letting $\mu = E(Y), \sigma^2 = Var(Y), \nu = E(X)$ and $K = (C + \nu)/2$, the following concentration of measure inequalities hold for all $t > 0$,*

$$P\left(\frac{Y-\mu}{\sigma} \geq t\right) \leq \begin{cases} \exp\left(-\frac{t^2\sigma^2}{2K\mu}\right) & \text{for } t \in [0, \gamma K\mu/\sigma^2) \\ \exp\left(-\gamma t + \frac{K\mu\gamma^2}{2\sigma^2}\right) & \text{for } t \in [\gamma K\mu/\sigma^2, \infty), \end{cases} \qquad \text{and } P\left(\frac{Y-\mu}{\sigma} \leq -t\right) \leq \exp\left(-\frac{t^2\sigma^2}{2\nu\mu}\right).$$

Proof. The proof is similar to that of Theorem 5.1. Since $Y^s = Y + X$ with Y and X independent and $X \geq 0$, using (5) with $\theta \in (0, \gamma)$ we have,

$$\begin{aligned} E(e^{\theta Y^s} - e^{\theta Y}) &= E(e^{\theta(X+Y)} - e^{\theta Y}) \leq \frac{1}{2}E\left(\theta X(e^{\theta(X+Y)} + e^{\theta Y})\right) \\ &= \frac{\theta}{2}E\left(X(e^{\theta X} + 1)e^{\theta Y}\right) = \frac{\theta}{2}E\left(X(e^{\theta X} + 1)\right)E(e^{\theta Y}) \\ &\leq \frac{\theta}{2}(E(Xe^{\gamma X}) + E(X))E(e^{\theta Y}) \\ &= K\theta m(\theta) \quad \text{where } K = (C+\nu)/2 \text{ and } m(\theta) = E(e^{\theta Y}). \end{aligned}$$

Now adding $m(\theta)$ to both sides yields

$$E(e^{\theta Y^s}) \leq (1 + K\theta)m(\theta),$$

and therefore

$$m'(\theta) = E(Ye^{\theta Y}) = \mu E(e^{\theta Y^s}) \leq \mu(1 + K\theta)m(\theta). \tag{55}$$

Again, with $M(\theta)$ the moment generating function of $(Y - \mu)/\sigma$,

$$M(\theta) = Ee^{\theta(Y-\mu)/\sigma} = e^{-\theta\mu/\sigma}m(\theta/\sigma),$$

by (55) we have,

$$\begin{aligned} M'(\theta) &= -(\mu/\sigma)e^{-\theta\mu/\sigma}m(\theta/\sigma) + e^{-\theta\mu/\sigma}m'(\theta/\sigma)/\sigma \\ &\leq -(\mu/\sigma)e^{-\theta\mu/\sigma}m(\theta/\sigma) + (\mu/\sigma)e^{-\theta\mu/\sigma}\left(1 + K\frac{\theta}{\sigma}\right)m(\theta/\sigma) \\ &= (\mu/\sigma^2)K\theta M(\theta). \end{aligned} \tag{56}$$

Integrating, and using the fact that $M(0) = 1$ yields

$$M(\theta) \leq \exp\left(\frac{K\mu\theta^2}{2\sigma^2}\right) \quad \text{for } \theta \in (0, \gamma).$$

Hence for a fixed $t > 0$, for all $\theta \in (0, \gamma)$,

$$P\left(\frac{Y-\mu}{\sigma} \geq t\right) \leq e^{-\theta t} M(\theta) \leq \exp\left(-\theta t + \frac{K\mu\theta^2}{2\sigma^2}\right).$$

The infimum of the quadratic in the exponent is attained at $\theta = t\sigma^2/K\mu$. When this value lies in $(0, \gamma)$ we obtain the first, right tail bound, for t in the bounded interval, while setting $\theta = \gamma$ yields the second.

Moving on to the left tail bound, using (5) for $\theta < 0$ yields

$$E(e^{\theta Y} - e^{\theta Y^s}) \quad \leq \quad -\frac{\theta}{2} E((Y^s - Y)(e^{\theta Y} + e^{\theta Y^s})) \leq -\theta E(X e^{\theta Y}) = -\theta E(X) E(e^{\theta Y}).$$

Rearranging we obtain

$$m'(\theta) = \mu E(e^{\theta Y^s}) \geq \mu(1 + \theta\nu) m(\theta).$$

Following calculations similar to (56) one obtains

$$M'(\theta) \geq (\mu/\sigma^2)\nu\theta M(\theta) \quad \text{for all } \theta < 0,$$

which upon integration over $[\theta, 0]$ yields

$$M(\theta) \leq \exp\left(\frac{\nu\mu\theta^2}{2\sigma^2}\right) \quad \text{for all } \theta < 0.$$

Hence for any fixed $t > 0$, for all $\theta < 0$,

$$P\left(\frac{Y-\mu}{\sigma} \leq -t\right) \leq e^{\theta t} M(\theta) \leq \exp\left(\theta t + \frac{\nu\mu\theta^2}{2\sigma^2}\right). \tag{57}$$

Substituting $\theta = -t\sigma^2/(\nu\mu)$ in (57) yields the lower tail bound, thus completing the proof. □ □

Though Theorem 5.2 applies in principle to all nonnegative infinitely divisible distributions with generating functions for Y and X that satisfy the given growth conditions, we now specialize to the subclass of compound Poisson distributions, over which it is always possible to determine the independent increment X. Not too much is sacrificed in narrowing the focus to this case, since a nonnegative infinitely divisible random variable Y has a compound Poisson distribution if and only if $P(Y = 0) > 0$.

5.2.2 Compound Poisson distribution

One important subfamily of the infinitely divisible distributions are the compound Poisson distributions, that is, those distributions that are given by

$$Y = \sum_{i=1}^{N} Z_i, \quad \text{where } N \sim \text{Poisson}(\lambda), \text{ and } \{Z_i\}_{i=1}^{\infty} \text{ are independent and distributed as } Z. \tag{58}$$

Compound Poisson distributions are popular in several applications, such as insurance mathematics, seismological data modelling, and reliability theory; the reader is referred to [3] for a detailed review.

Although Z is not in general required to be nonnegative, in order to be able to size bias Y we restrict ourselves to this situation. It is straightforward to verify that when the moment generating function $m_Z(\theta) = Ee^{\theta Z}$ of Z is finite, then the moment generating function $m(\theta)$ of Y is given by

$$m(\theta) = \exp(-\lambda(1 - m_Z(\theta))).$$

In particular $m(\theta)$ is finite whenever $m_Z(\theta)$ is finite. As Y in (58) is infinitely divisible the equality (54) holds for some X; the following lemma determines the distribution of X in this particular case.

Lemma 5.1. *Let Y have the compound Poisson distribution as in (58) where Z is nonnegative and has finite, positive mean. Then*

$$Y^s = Y + Z^s,$$

has the Y size biased distribution, where Z^s has the Z size bias distribution and is independent of N and $\{Z_i\}_{i=1}^{\infty}$.

Proof. Let $\phi_V(u) = Ee^{iuV}$ for any random variable V. If V is nonnegative and has finite positive mean, using $f(y) = e^{iuy}$ in (1) results in

$$\phi_{V^s}(u) = \frac{1}{EV}\left(EVEe^{iuV^s}\right) = \frac{1}{EV}EVe^{iuV} = \frac{1}{iEV}\phi_V'(u). \tag{59}$$

It is easy to check that the characteristic function of the compound Poisson Y in (58) is given by

$$\phi_Y(u) = \exp(-\lambda(1 - \phi_Z(u))), \tag{60}$$

and letting $EZ = \vartheta$, that $EY = \lambda\vartheta$. Now applying (59) and (60) results in

$$\phi_{Y^s}(u) = \frac{1}{i\lambda\vartheta}\phi_Y'(u) = \frac{1}{i\vartheta}\phi_Y(u)\phi_Z'(u) = \phi_Y(u)\phi_{Z^s}(u).$$

□ □

To illustrate Lemma 5.1, consider the Cramér-Lundberg model [10] from insurance mathematics. Suppose an insurance company starts with an initial capital u_0, and premium is collected at the constant rate α. Claims arrive according to a homogenous Poisson process $\{N_\tau\}_{\tau\geq 0}$ with rate λ, and the claim sizes are independent with common distribution Z. The aggregate claims Y_τ made by time $\tau \geq 0$ is therefore given by (58) with N and λ replaced by N_τ and λ_τ, respectively.

Distributions for Z which are of interest for applications include the Gamma, Weibull, and Pareto, among others. For concreteness, if $Z \sim \text{Gamma}(\alpha, \beta)$ then $Z^s \sim \text{Gamma}(\alpha+1, \beta)$, and the mean ν of the increment Z^s, and the mean μ_τ and variance σ_τ^2 of Y_τ, are given by

$$\nu = \quad (\alpha + 1)\beta, \quad \mu_\tau = \lambda\tau\alpha\beta \quad \text{and} \quad \sigma_\tau^2 = \lambda\tau\beta^2\alpha.$$

The conditions of Theorem 5.2 are satisfied with any $\gamma \in (0, 1/\beta)$ since $E(e^{\theta Y}) < \infty$ and $E(Z^s e^{\theta Z^s}) < \infty$ for all $\theta < 1/\beta$. Taking $\gamma = 1/(M\beta)$ for $M > 1$ for example, yields

$$C = E(Z^s e^{\gamma Z^s}) = (\alpha + 1)\beta(\frac{M}{M-1})^{\alpha+2}.$$

For instance, the lower tail bound of Theorem 5.2 now yields a bound on the probability that the aggregate claims by time τ will be 'small', of

$$P\left(\frac{Y_\tau - \mu_\tau}{\sigma_\tau} \leq -t\right) \leq \exp\left(-\frac{t^2}{2(\alpha+1)}\right).$$

It should be noted that in some applications one may be interested in Z which are heavy tailed, and hence do not satisfy the conditions in Theorem 5.2.

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